Reasoning with Diagrams: Observation, Inference and Overspecificity

Gem Stapleton

Centre for Secure, Intelligent and Usable Systems,
University of Brighton, UK
g.e.stapleton@brighton.ac.uk

Abstract

The ability of diagrams to convey information effectively comes, in part, from their ability to make facts explicit that would otherwise need to be inferred. This type of advantage has often been referred to as a free ride and was deemed to occur only when a diagram was obtained by translating a symbolic representation of information. Recent work generalised free rides, introducing the idea of an observational advantage, where the existence of such a translation is not required. In this paper, I will provide an overview of the theory of observation. It has been shown that Euler diagrams without existential import have significant observational advantages over set theory: they are observationally complete. I will then explore to what extent Euler diagrams with existential import are observationally complete with respect to set-theoretic sentences. In particular, has been shown that existential import significantly limits the cases when observational completeness arises, due to the potential for overspecificity. These two results formally support Larkin and Simon's claim that "a diagram is (sometimes) worth ten thousand words". The work in this invited paper is derived from previously published results as cited in the text.

1. Introduction

The choice of notation in which to represent information is an important consideration if the desire is for effective communication. But, even when a choice has been made, one must still select from the semantically equivalent representations of the information to be conveyed. Understanding the impact of such choices from the perspective of human cognition is important. This paper presents an overview of selected state-of-the-art work on these choices from a theoretical perspective, summarising results previously published in [12, 13].

There are many ways in which visual representations can be manipulated to impact their effectiveness. For instance, graphical features, such as colour or size, can be manipulated to enhance or diminish the effectiveness of a representation [2]. The particular focus of this paper is the recent work on *observational advantages* [12], which generalises prior work on *free rides* [9]. If we can understand when and how one representation of information has observational advantages over another then it allows us to make an informed choice of representation: we should choose a representation with many observational advantages.

An example can be seen in Figure 1. Here, five textual statements convey information about countries visited by people. They are translated into an Euler diagram which represents exactly the same information. For instance, the first textual statement is visualized by the curve labelled Germany being drawn inside the curve labelled Italy. The fifth statement is shown by the fact that the curves labelled Qatar and Sudan do not overlap. From the diagram, one can simply read off that everyone who visited Germany visited Qatar but this fact needs to be *inferred* from the text. As such, 'everyone who visited Germany visited Qatar' is a free ride from the diagram given the text.

Euler diagrams are not the only notation in which free rides arise. Figure 2 shows the same textual statements but this time translated into a semantically equivalent linear diagram. Here, lines are used to represent sets and their positions relative to each other along the x-axis provides information about subset and disjointness relationships. So, the information that no one visited both Qatar and Sudan is shown by the fact that the two corresponding lines do not overlap. This diagram has exactly the same free rides, given the associated text, as the Euler diagram in Figure 1. For instance, since the lines for Germany and Sudan do not overlap, we can read off that no one visited both Germany and Sudan, which again must be inferred from the text.

Free rides, in general, are defined to arise only when one representation of information is derived by systematically translating another, given, representation of information (see [12] for details). By contrast, observational advan-

Everyone who visited Germany visited Italy
Everyone who visited Germany visited Mali
Everyone who visited Italy visited Qatar
Everyone who visited Mali visited Qatar
No one visited both Qatar and Sudan
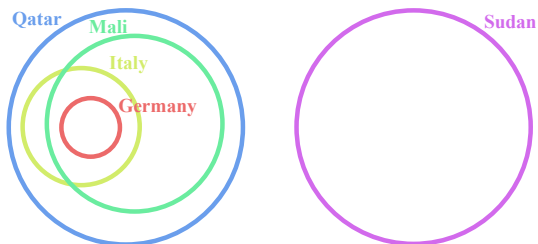
Figure 1. Illustrating free rides in Euler diagrams.

Everyone who visited Germany visited Italy
Everyone who visited Germany visited Mali
Everyone who visited Italy visited Qatar
Everyone who visited Mali visited Qatar
No one visited both Qatar and Sudan
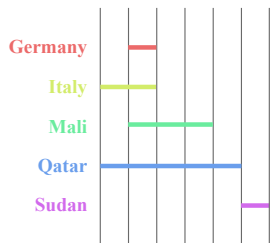
Figure 2. Illustrating free rides in linear diagrams.

tages, of which free rides are examples, only require that the two representations are semantically equivalent, removing the need for a translation. Both of them capture the idea that information which can simply be read off from one representation but must be inferred from another can be considered a (potential) advantage of the former over the latter.

In section 2 we discuss, in more detail, the role of meaning-carriers and observation, where we look at observational advantages. Section 3 demonstrates, by example, how Euler diagrams without existential import (like the examples we have seen so far) are what is known to be *observationally complete* with respect to set-theoretic statements. When the existential import assumption is introduced, observational completeness is no longer guaranteed to hold, demonstrated in section 4. We then conclude in section 5, where we highlight the need for future work, specifically fo-

cusing on cognition and usability. The main results in this invited paper are derived from previously published results, primarily [11, 13].

2. Observation

A formal account of the idea of an observational advantage and, therefore, of free rides, requires a formal definition of the syntax and semantics of the notations being considered. Of course, we do not have that for natural language in general, although the statements given in Figures 1 and 2 are clearly in a controlled form. To ease our exposition, therefore, we consider versions of these statements in set-theoretic notation: $Germany \subseteq Italy$, $Germany \subseteq Mali$, $Italy \subseteq Qatar$, $Mali \subseteq Qatar$, and $Qatar \cap Sudan = \emptyset$. Each of these five statements[1] express the desired information. What is important here is that each statement has a single *meaning-carrying relationship*.

This idea of a meaning-carrying relationship is at the heart of how free rides and observational advantages are defined. So what is a meaning-carrying relationship? Well, it is taken to be a relation on the syntax of a representation that evaluates to either 'true' or 'false' when the syntax is given meaning. Therefore, a meaning-carrier is similar to a 'representing fact' in Shin's work [10]. Why, then, do each of the five set-theoretic statements have a single meaning-carrying relationship? Consider $Germany \subseteq Italy$, which is either true or false, depending on the sets represented by $Germany$ and $Italy$. The meaning-carrying relationship is that $Germany$ is written to the left of $\subseteq$ and $Italy$ to the right. Similarly, the fifth statement's meaning carrier is that $Qatar \cap Sudan$ is written to the left of $=$ and $\emptyset$ is on the right.

What, then, are the meaning-carrying relationships in the Euler and linear diagrams? Each diagram is a single statement but they have *many* meaning-carrying relationships. The Euler diagram in Figure 1 uses curve containment to express the four subset statements. For the two curves arising from $Germany \subseteq Italy$, we see that the one is inside the other and the assertion made by the diagram is true whenever the set represented by $Germany$ is a subset of that represented by $Italy$, otherwise it is false. So, one example of a meaning-carrying relationship is the containment of one curve by another. Likewise, the Euler diagram uses non-overlapping curves to express the disjointness of sets.

In addition to the five meaning-carrying relationships corresponding to the five textual statements, we have further meaning-carrying relationships, such as the non-overlapping of the $Mali$ and $Sudan$ curves which expresses that no one visited both $Mali$ and $Sudan$. Thus, from the

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[1]A statement is a syntactic entity (in any representation system) that represents some information. For example, a set-theoretic sentence is a single statement, and so is an Euler diagram.

Euler diagram we can *observe* that $Mali \cap Sudan = \emptyset$ but we must infer this from the set-theoretic statements. In the linear diagram, Figure 2, the statements that can be observed are the same as those that can be observed from the Euler diagram, even though their syntax is different and their meaning-carriers are therefore identified in a different way.

The presence of multiple meaning-carrying relationships suggests that, as compared to representation systems with single-meaning carrying relationship, facts can sometimes be observed to be true rather than inferred to be true. Representations of information that allow statements to be observed as true, without the need for inference, can be considered advantageous. Here we must be clear, though, that this difference between single and multiple meaning carriers should not suggest a dichotomy of sentential and diagrammatic notations. Merely, the examples we have presented contrast symbolic and diagrammatic representations where the former has single meaning-carriers and the latter has multiple meaning-carriers.

So, we have seen that meaning-carriers can lead to information being observed as true from representations of information. The concept of observation has been considered in proof systems, where it was formalized as an inference rule [1, 14]. To give the idea, an inference rule based on observation allows one to identify pieces of information expressed in a statement and re-express them in another statement. As we have already seen, many sentences support precisely one meaning-carrying relationship. In such cases, if we were to define and apply an observation inference rule, it would merely restate the information in an identical, or semantically equivalent sentence. By contrast, in systems where single statements have many meaning-carriers, such as some diagrammatic representations, using meaning-carriers to apply an observation inference rule can yield many different statements. This suggests that the role of observation is important when considering inference problems.

One example where observation has been used is Barwise and Etchemendy's *Hyperproof* system [1]. Moreover, Swoboda and Allwein [14] included both an observation rule and other inference rules in their work. They called for the distinctive treatment of the observation rule, which consists of visual perception and the restatement of the information thus obtained. This draws on Dretske's classification of various cases that are commonly described as "somebody's seeing that something is the case" [5].

It is necessary to define meaning-carriers and the notion of observation in the context of the syntax and semantics of notations; what it means to be a meaning-carrier is clearly determined by the syntax and semantics. However, if we assume such definitions are given, we can proceed define what it means to be an observational advantage. Firstly, we require that if we observe a statement, σ_o, from another statement, σ then the following properties must hold:

1. some of the meaning-carrying relationships holding in σ also hold in σ_o, and

2. σ_o supports just enough relationships to express the meanings carried by the selected relationships in σ and nothing stronger [12].

These properties ensure that σ_o is semantically entailed by σ.

Suppose now that we have the more general case of a set of statements, Σ, from which we wish to observe information: the *only* meaning-carrying relationships in Σ are derived from the statements in Σ. So, the only statements observable from Σ must be observable from one of the statements in Σ. This leads us to be able to define the notion of observation from a set of statements.

Definition 1 *Let Σ be a finite set of statements and σ_o be a single statement. Then σ_o is **observable** from Σ iff σ_o is observable from some statement, σ, in Σ. The set of statements that are observable from Σ is denoted $\mathcal{O}(\Sigma)$ [12].*

Now we have understood what it means to be observable, we can define what it means to be an observational advantage. Intuitively, an observational advantage occurs when we have two semantically equivalent representations of information, say two sets of statements, Σ and $\hat{\Sigma}$. If we can observe a statement from $\hat{\Sigma}$ but not from Σ then it is an advantage of $\hat{\Sigma}$ over Σ. This is captured by definition 2.

Definition 2 *Let Σ and $\hat{\Sigma}$ be finite, semantically equivalent sets of statements. Let σ be a statement. If*

1. *σ is not observable from Σ, and*

2. *σ is observable from $\hat{\Sigma}$*

*then σ is an **observational advantage** of $\hat{\Sigma}$ given Σ. The set of all observational advantages of $\hat{\Sigma}$ given Σ is denoted $\mathcal{OA}(\hat{\Sigma}, \Sigma)$ [12].*

To finish this section, we consider two extreme cases where observational advantages can arise, or even fail to do so. Firstly, suppose given a set of statements, Σ, there is some information, captured by a set, $\Sigma_\models$, of statements, whose truth we want to establish. If we can simply read-off (observe) *all* of the respective statements in $\Sigma_\models$ then Σ can be considered *observationally complete*:

Definition 3 *Let Σ and $\Sigma_\models$ be finite sets of statements. Then Σ is **observationally complete** with respect to $\Sigma_\models$ if*

$$\Sigma_\models \subseteq \mathcal{O}(\Sigma) \text{ [12].}$$

Lastly, we have the other extreme case: if we cannot simply read-off (observe) *any* of the respective statements, then Σ can be considered *observationally devoid*:

Definition 4 *Let Σ and $\Sigma_\vDash$ be finite sets of statements. Then Σ is **observationally devoid** with respect to $\Sigma_\vDash$ if*

$$\Sigma_\vDash \cap \mathcal{O}(\Sigma) = \emptyset \ [12].$$

These two extreme cases allow us to formally establish when one representation has numerous advantages over another. This occurs when, say, Σ is observationally complete, yet $\hat{\Sigma}$ is observationally devoid with respect to a given $\Sigma_\vDash$.

3. Observational Advantages of Euler Diagrams without Existential Import

Euler diagrams have substantial observational advantages over set-theoretic *statements*. For our purposes, we consider set-theoretic *expressions* of the form $s_1 \cap s_2$, $s_1 \cup s_2$, $s_1 \backslash s_2$ and $\overline{s_1}$ as well as the special symbols U (the universal set) and $\emptyset$ (the empty set). We can then form set-theoretic statements using these expressions. The following are the set-theoretic statements that we consider: $s_1 \subseteq s_2$ and $s_1 = s_2$. This allows a rich variety of information about sets to be expressed. What is important here is that given any finite collection of set-theoretic statements, there exists a semantically equivalent Euler diagram (without existential import).

As an example, consider the following set-theoretic statements:

1. no one visited both Sudan and Vietnam:

$$Sudan \cap Vietnam = \emptyset$$

2. no one visited both Denmark and Vietnam:

$$Denmark \cap Vietnam = \emptyset$$
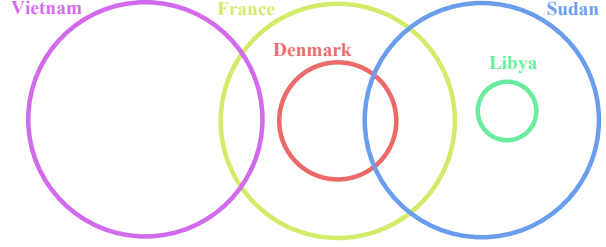
3. everyone who visited Denmark visited France:

$$Denmark \subseteq France$$

4. everyone who visited Libya visited Sudan:

$$Libya \subseteq Sudan$$

5. no one visited both France and Libya:

$$France \cap Libya = \emptyset.$$

Figure 3. Observational completeness.

A semantically equivalent Euler diagram (without existential import) can be seen in Figure 3.

We have already seen some examples of set-theoretic statements that are observable from Euler diagrams. In this particular case, one observable statement is

$$Vietnam \cap Libya = \emptyset$$

as well as the equivalent statement $Libya \cap Vietnam = \emptyset$. More complex set-theoretic statements are also observable, such as

$$(Denmark \cup Libya) \cap Vietnam = \emptyset.$$

Why can this more complex statement be observed? Well, here we must consider the regions formed by the curves. The region, comprising three *zones*, formed by the interiors of the curves for Denmark and Libya is disjoint from (shares no points with) the region inside the Vietnam curve. Thus, the *disjointness* of two regions is a meaning-carrier in Euler diagrams. The presence of this meaning-carrier shows that we can observe $(Denmark \cup Libya) \cap Vietnam = \emptyset$ from the diagram.

Consider, then, the types of set-theoretic expressions to which we have access, in the context of our example:

1. Each 'basic' set-expression (i.e. one that does not involve intersection, union, difference or complement) corresponds to a region inside a curve. In our example, there are five basic set-expressions: $Denmark$, $France$, $Libya$, $Sudan$, and $Vietnam$. We say, informally, that these basic set-expressions *correspond* to regions in the diagram.

2. Given any two set–expressions, s_1 and s_2, that correspond to regions in the diagram, the set-expressions $s_1 \cap s_2$, $s_1 \cup s_2$, $s_1 \backslash s_2$, and $\overline{s_1}$ also correspond to regions in the diagram (noting that 'empty' regions contain no points). For example, $Denmark \cup France$, $Denmark \cap France$, $Sudan \backslash France$ and $\overline{Libya}$ all correspond to regions.

Using this insight, it is easy to see that any set-expression that can be formed from the basic ones corresponds to some

region in the diagram. Therefore, from the diagram, given any set-theoretic statement of the form $s_1 \subseteq s_2$, there are corresponding regions for s_1 and s_2. We can the *observe* $s_1 \subseteq s_2$ from the diagram precisely when the region, r_1 for s_1 is a subset (contained by) the region for s_2. Likewise, if $r_1 = r_2$ then we can observe $s_1 = s_2$. This insight leads us to the following theorem, which is derived from one of the main results in [12]:

Theorem 1 *Let $\mathcal{S} = \{s_1, ..., s_n\}$ be a finite set of set-theoretic statements. Then there exists an Euler diagram, d, where:*

1. *d semantically equivalent to S, and*

2. *given any set-theoretic statement, s, that is formed from set-expressions whose basic sets are all used in S and which is semantically entailed by S can be observed from d.*

In other words, d is *observationally complete* with respect to the set, $S_\vDash$, of all statements that we can infer from S. But, on the other hand, what can be observed from S? Well, since the set-theoretic statements in S only have one meaning carrier, S is observationally devoid, given $S_\vDash \backslash S$. This means that the Euler diagram has maximal observational advantage over S and these result tells us that Euler diagrams are powerful representations of information compared to set-theoretic statements.

4. Observational Advantages of Euler Diagrams with Existential Import

Euler diagrams that do not enforce existential import, which have been the focus up to this point, allow regions in the diagram to represent empty sets. For instance, in Figure 3, there is no information that anyone at all visited Vietnam (or the other countries). There are occasions, of course, when we want to enforce the non-emptiness of sets (or, more generally, provide cardinality information, but that is beyond the scope of this discussion). Euler diagrams can be extended in various was to achieve this.

For instance, Peirce denotes the non-emptiness of a set with $\otimes$-sequences [8], also used by Shin [10] and further developed by Choudhury and Chakraborty [4]. By contrast, Euler diagrams with existential import [3] do not require additional syntax to assert the non-emptiness of a set: all zones are taken to represent non-empty sets (so, any region formed by the curves represents a non-empty set) [6]. The extension of the semantics to require that zones represent non-empty sets leads to an obvious question: are Euler diagrams with existential import still observationally complete?

To answer this question, we need to consider a wider variety of set-theoretic statements, even though the use of

intersection, union, difference and complement for forming set-expressions is still appropriate and sufficient. This is, obviously, because using just $\subseteq$ and $=$ does not lead to assertions about non-emptiness. Therefore, we also allow the use of $\nsubseteq$ and $\neq$. Using these two additional operators, we can form statements like $Vietnam \neq \emptyset$ and $Denmark \nsubseteq Sudan$ (so implying that $Denmark \backslash Sudan \neq \emptyset$).
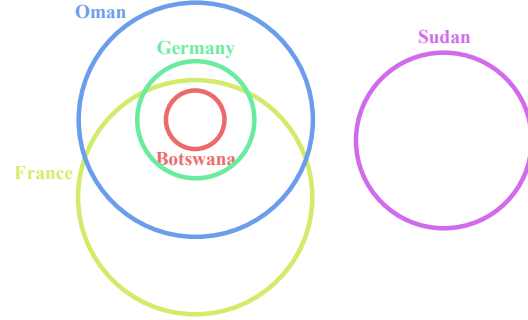
Figure 4. An Euler diagram interpreted under existential import semantics.

Consider, now, the Euler diagram in Figure 4 under the existential import assumption. Can we find a set of set-theoretic sentences that capture the meaning conveyed by this diagram? There are four statements corresponding to the set-theoretic relationships expressible using $\subseteq$ and $=$ (noting that other statements could be written down too, but they would not convey extra subset or equality information):

1. everyone who visited Botswana visited France and Germany:

$$Botwana \subseteq France \cap Germany,$$

2. everyone who Germany visited Oman:

$$Germany \subseteq Oman,$$

3. no one visited both Oman and Sudan:

$$Oman \cap Sudan = \emptyset,$$

and

4. no one visited both France and Sudan:

$$France \cap Sudan = \emptyset.$$

However, these statements alone are not semantically equivalent to the Euler diagram under the existential import assumption, since each zone represents a non-empty set. We need a further seven statements to achieve semantic equivalence (again, different choices of statements exist - there are non-unique sets of set-theoretic statements that are semantically equivalent to the diagram):

1. at least one person visited Oman only:

$$Oman \cap \overline{Botswana} \cap \overline{France} \cap \overline{Germany} \cap \overline{Sudan} \neq \emptyset,$$

2. at least one person visited France only:

$$France \cap \overline{Botswana} \cap \overline{Germany} \cap \overline{Oman} \cap \overline{Sudan} \neq \emptyset,$$

3. at least one person visited Sudan only:

$$Sudan \cap \overline{Botswana} \cap \overline{France} \cap \overline{Germany} \cap \overline{Oman} \neq \emptyset,$$

4. at least one person visited both Germany and Oman, but no other country:

$$Germany \cap Oman \cap \overline{Botswana} \cap \overline{France} \cap \overline{Sudan} \neq \emptyset,$$

5. at least one person visited both France and Oman, but no other country:

$$France \cap Oman \cap \overline{Botswana} \cap \overline{Germany} \cap \overline{Sudan} \neq \emptyset,$$

6. at least one person visited all of France, Germany and Oman, but no other country:

$$France \cap Germany \cap Oman \cap \overline{Botswana} \cap \overline{Sudan} \neq \emptyset,$$

   and

7. at least one person visited all of Botswana, France, Germany and Oman:

$$Botswana \cap France \cap Germany \cap Oman \neq \emptyset,$$

Notice that the last statement does not involve all five sets. This is sufficient because we know that the set specified by the four-way intersection is disjoint from Sudan (which follows from the earlier statements). We could, instead, have chosen the set-theoretic sentence

$$Botswana \cap France \cap Germany \cap Oman \cap \overline{Sudan} \neq \emptyset.$$

This illustrates how choice of statements can arise, when seeking semantic equivalence.

As with Euler diagrams without existential import, we can observe information from the diagram that needs to be inferred from the set-theoretic statements. For instance, any region represents a non-empty set, so we can observe that $Oman \neq \emptyset$ and $France \cap Oman \neq \emptyset$. We can further observe that $Oman \nsubseteq France$. Many other statements can be observed too. In fact, this diagram is observationally complete with respect to the set of set-theoretic sentences that are given in the two lists above.

What should be evident from this example is that given an arbitrary set of set-theoretic statements there need not exist a semantically equivalent Euler diagram. For instance, the first five statements, focusing on subset and equality, cannot be translated into a semantically equivalent Euler diagram with existential import. If we were to attempt to produce such diagram it would be that in Figure 4, but we know that this expresses more information than those five statements alone.

The problem here arises because of overspecificity. Unfortunately, due to overspecificity, there are numerous sets of set-theoretic sentences where no semantically equivalent Euler diagram with existential import exists. This indicates a problematic situation: diagrams are typically believed to excel as representations of information due to their ability to make facts explicit that would otherwise need to be inferred. But, as a positive, what we can take away from this discussion is that, given a finite set of set-theoretic sentences, if there exists a semantically equivalent Euler diagram with existential import then that diagram is observationally complete [13][2].

5. Conclusion

It has been seen, though a consideration of Euler diagrams under varying semantic conventions, that sometimes they are capable of representing information in an observationally complete way. The incorporation of existential import brings with it increased expressiveness but at a price: overspecificity means that often information cannot be expressed appropriately. The results support Larkin and Simon's claim that "a diagram is (sometimes) worth ten thousand words" [7].

The theoretical characterisation of what it means to be observable, via meaning-carriers, and the subsequent consideration of observational completeness, is driven by a desire to understand what makes diagrams cognitively more effective than symbolic or textual representations. Now that such theory has been developed, it is important to conduct empirical studies to ascertain the relationship between observational advantages and cognitive advantages. Are observational advantages really helpful? That is, do people extract information more effectively from a representation with an observational advantage over another, or does the process of inference lead to more effective task performance? Assuming that observational advantages do bring cognitive benefit, is there a way of modelling the relative cognitive benefit of some observational advantages over others? This latter question is inspired by the fact that some set-theoretic statements are likely to be more readily observable, by people, from a diagram than others.

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[2] The conditions under which this happens are non-trivial and so omitted.

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